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ZERO DIVISOR GRAPHS OF SKEW HURWITZ SERIES RINGS

KAMAL PAYKAN - HOSSEIN DANESHMAND

For a ring endomorphism α, we investigate the interplay between the ring-theoretical properties of the skew Hurwitz series ring (HR,α) and the graph-theoretical properties of its zero-divisor graph $\overline{\Gamma}((HR,\alpha))$. Furthermore, we examine the preservation of diameter and girth of the zero-divisor graph under extension to skew Hurwitz series rings.

1. Introduction

The concept of a zero-divisor graph of a commutative ring was introduced by Beck in [7]. In his work all elements of the ring were vertices of the graph (see also [3]). In [4], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors of a ring. This graph turns out to best exhibit the properties of the set of zero-divisors of a commutative ring. In [30], Redmond studied the zero-divisor graph of a non-commutative ring. Several papers are devoted to studying the relationship between the zero-divisor graph and algebraic properties of rings (cf.[2], [3], [5], [7], [30], [34]).

The zero-divisors of R, denoted by $Z(R)$, is the set of elements $a \in R$ such that there exists a non-zero element $b \in R$ with $ab = 0$ or $ba = 0$. Let $Z^*(R)$

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Corresponding author: Kamal Paykan

denote the (nonempty) set of nonzero zero divisors. The directed graph $\Gamma(R)$ is a graph with vertices in $Z^*(R)$, where $x \to y$ is an edge between distinct vertices x and y if and only if $xy = 0$. Recently Redmond in [30] has extended this concept to any arbitrary ring. Redmond in [30] defined an undirected zero-divisor graph of a non-commutative ring R, denoted by $\overline{\Gamma}(R)$, with vertices in the set $Z(R)^*$ and such that two distinct vertices a and b are adjacent if and only if $ab = 0$ or $ba = 0$. Note that for a commutative ring R, the definition of the zero-divisor graph of R in [4] coincides with the definition of $\overline{\Gamma}(R)$.

According to Cohn [10], a ring R is called *reversible* if $ab = 0$ implies that $ba = 0$ for $a, b \in R$. So, in view of [30, Theorem 2.3], over a reversible ring R, the simple (undirected) graph $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$, where $diam(\Gamma(R))$ is the diameter of $\Gamma(R)$. In [30] it has been shown that for any ring R, every two vertices in $\overline{\Gamma}(R)$ are connected by a path of length at most 3. Note that using the proof of this result in the commutative case, one can establish that for any arbitrary ring R, if there exists a path between two vertices x and y in the directed graph $\Gamma(R)$, then the length of the shortest path between x and y is at most 3. Moreover, in [30] it is shown that for any ring R, if $\overline{\Gamma}(R)$ contains a cycle, then the length of the shortest cycle in $\overline{\Gamma}(R)$ is at most 4. There is a considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions.

The zero-divisor graphs offer a graphical representation of rings so that we may discover some new algebraic properties of rings that are hidden from the viewpoint of classical ring theorists. For instance, using the notion of a zero-divisor graph, it has been proven in [31] that for any finite ring R, the sum $\sum_{x\in R} || r_R(x) | - | \ell_R(x) ||$ is even, where $r_R(x)$ and $\ell_R(x)$ denote the right and left annihilators of the element x in R, respectively. More recently, Axtell, Coykendall and Stickles, in [6], examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings. Also, Lucas, in [23], continued the study of the diameter of polynomial and power series of commutative rings. Moreover, Anderson and Mulay, in [5], studied the girth and diameter of a commutative ring and investigated the girth and diameter of polynomial and power series of commutative rings. For a commutative ring R with a monomorphism α and an α-derivation δ, Afkhami, Khashyarmanesh and Khorsandi, in [1], compare the diameter (and girth) of the zero-divisor graphs of R and the Ore extension $R[x;\alpha,\delta]$, when $R[x;\alpha,\delta]$ is assumed to be reversible.

In [25], the authors studied the interaction between the ring-theoretical properties of a skew generalized power series ring and the graph-theoretical properties of its zero-divisor graph. Motivated by results in [25], we examine the preservation and lack thereof of the diameter and girth of the zero-divisor graph of a non-commutative ring under extension to *skew Hurwitz series ring construction* (HR,α), where R is a ring and α is an endomorphism of R (the definition of the ring (HR,α) will be recalled in Section 2). In Section 3, we prove that if R is an α-compatible ring which is torsion free as a $\mathbb{Z}$-module, then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}((HR,\alpha))$ is complete. Furthermore, we compare the diameter (and girth) of the zero-divisor graphs $\overline{\Gamma}(R)$ and $\overline{\Gamma}((HR,\alpha))$. Finally we give a complete characterization for the girth of $\overline{\Gamma}((HR,\alpha))$.

For two distinct vertices a and b in the simple (undirected) graph Γ, the *distance* between a and b, denoted by $d(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise we put $d(a,b) := \infty$. The *diameter* of a graph Γ is $diam(\Gamma) := sup\{d(a,b) \mid a \text{ and } b \text{ are distinct vertices of } \Gamma\}$. The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. The *girth* of a simple (undirected) connected graph Γ, denoted by $gr(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise $gr(\Gamma) := \infty$. Also, we use A^* to denote the nonzero elements of A, and $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Z}_n$ for the integers, positive integers and the integers modulo n, respectively. For a nonempty subset X of a ring R, $r_R(X)$ (resp. $\ell_R(X)$) is used for the right (resp. left) annihilator of X over R.

2. Preliminaries

Rings of formal power series have been of interest and have had important applications in many areas, one of which has been differential algebra. In an earlier paper by Keigher [16], the ring of Hurwitz series, a variant of the ring of formal power series, was considered, and some of its properties, especially its categorical properties, were studied. In the papers [17], [18] Keigher demonstrated that the ring of Hurwitz series has many interesting applications in differential algebra and in the discussion about weak normalization. Its product, a product of sequences using binomial coefficients, was studied in papers by Fleiss [11] and Taft [32]. While there are many studies of these rings over a commutative ring, very little is known about them over a noncommutative ring. The ring-theoretical properties of skew Hurwitz series rings have been investigated by many authors (see [16], [17], [18], [21], [27], [28] and [29]). In the present paper we study Hurwitz series over a noncommutative ring with identity, examine its structure and properties.

Throughout this paper, R denotes an associative ring with unity and $\alpha : R \to R$ is an endomorphism such that $\alpha(1) = 1$. The ring (HR,α) of *skew Hurwitz series over a ring R* is defined as follows: the elements of (HR,α) are functions $f : \mathbb{N} \to R$, where $\mathbb{N}$ is the set of integers greater or equal than zero.

The operation of addition in (HR,α) is componentwise and the operation of multiplication is defined, for every $f,g\in(HR,\alpha)$, by:

$$fg(n)=\sum_{k=0}^{n}\begin{pmatrix} n\\ k\end{pmatrix} f(k)\alpha^{k}(g(n-k)) \text{ for each } n\in\mathbb{N},$$

where $\begin{pmatrix} n\\ k\end{pmatrix}$ is the binomial coefficient. In the case where the endomorphism α is the identity, we write HR instead of (HR,α). If one identifies a skew formal power series $\sum_{n=0}^{\infty} a_n x^n\in R[[x;\alpha]]$ with the function f such that $f(n)=a_n$, then multiplication in (HR,α) is similar to the usual product of skew formal power series, except that binomial coefficients appear in each term in the product introduced above. To each $r\in R$ and $n\in\mathbb{N}$, we associate elements $h_r,h_n^{'}\in(HR,\alpha)$ defined by

$$h_r(x)=\begin{cases} r & x=0\\ 0 & x\neq 0,\end{cases}\quad h_n^{'}(x)=\begin{cases} 1 & x=n\\ 0 & x\neq n.\end{cases}$$

It is clear that $r\mapsto h_r$ is a ring embedding of R into (HR,α) and also (HR,α) is a ring with identity h_1. For every nonempty subset X of R, we set:

$$(HX,\alpha)=\{f\in(HR,\alpha)\mid f(n)\in X\cup\{0\} \text{ for every } n\in\mathbb{N}\}.$$

Proposition 2.1. *Let R be a ring containing the field of rational numbers Q and α a left Q-algebra homomorphism of R. Then the rings (HR,α) and $R[[x;\alpha]]$ are isomorphic.*

Proof. We define a map $\psi:(HR,\alpha)\to R[[x;\alpha]]$, given by $\psi(f)=\sum_{n=0}^{\infty}\frac{f(n)}{n!}x^n$. It is easy to show that ψ is an isomorphism. □

Remark 2.2. Note that, since some of the results in this manuscript are already known for skew power series ring $R[[x;\alpha]]$, throughout the paper we assume that R is a ring not containing the field of rational numbers.

According to Krempa [15], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a)=0$ implies $a=0$, for $a\in R$. A ring R is said to be *α-rigid* if there exists a rigid endomorphism α of R.

Remark 2.3. Let I be an index set, D_i a domain for each $i\in I$, and $R=\prod_{i\in I} D_i$. Also assume that α_i is an endomorphism of D_i for each $i\in I$. Then we get an endomorphism α of R defined by the assignments $\alpha(\{r_i\}_{i\in I})=\{(\alpha_i)(r_i)\}_{i\in I}$. If α_i is injective for all $i\in I$, then R is an α-rigid ring.

In [13], the authors introduced α-compatible rings and studied their properties. A ring R is said to be *α-compatible* if for each $a,b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$. Basic properties of rigid and compatible endomorphisms, proved by Hashemi and Moussavi in [13, Lemmas 2.1 and 2.2], are summarized in the following lemma. It is used repeatedly in the sequel.

Lemma 2.4. *Let α be an endomorphism of a ring R. Then:*

(i) *if α is compatible, then α is injective;*

(ii) *α is compatible if and only if for all $a,b \in R$, $\alpha(a)b = 0 \Leftrightarrow ab = 0$;*

(iii) *the following conditions are equivalent:*

(1) *α is rigid;*

(2) *α is compatible and R is reduced;*

(3) *for every $a \in R$, $\alpha(a)a = 0$ implies that $a = 0$.*

In the proof of Theorems 3.1, 3.7, 3.9 and 3.14, we will need the following lemma.

Lemma 2.5. *Let R be an α-rigid ring such that it is torsion free as a $\mathbb{Z}$-module. Suppose that $f,g \in (HR,\alpha)$ are such that $fg = 0$. Then $f(n)g(m) = 0$ for all $n,m \in \mathbb{N}$.*

Proof. Since $fg = 0$, we obtain $f(0)g(0) = 0$, $f(0)g(1) + f(1)\alpha(g(0)) = 0$, $f(0)g(2) + 2f(1)\alpha(g(1)) + f(2)\alpha^2(g(0)) = 0$, $\ldots$. Now, we have $f(0)g(1) + f(1)\alpha(g(0)) = 0$, so multiplying from left by $g(0)$, we get $g(0)f(1)\alpha(g(0)) = 0$, since $g(0)f(0) = 0$. Thus $g(0)f(1)\alpha(g(0))\alpha(f(1)) = 0$. Since R is α-rigid ring, $g(0)f(1) = 0$. So $f(1)g(0) = 0$, and $f(1)\alpha(g(0)) = 0$, by Lemma 2.4. Hence $f(0)g(1) = 0$. Since $f(0)g(2)g(0) = 0$ and $f(1)\alpha(g(1))g(0) = 0$, we have $f(2)\alpha^2(g(0))g(0) = 0$. Thus $f(2)g(0) = 0$, by Lemma 2.4. So $f(0)g(2) + 2f(1)\alpha(g(1)) = 0$. Multiplying this equality on the left-hand side by $g(1)$, we get $g(1)f(0)g(2) + 2g(1)f(1)\alpha(g(1)) = 0$. Thus $2g(1)f(1)\alpha(g(1)) = 0$. Since R is α-rigid and torsion free as a $\mathbb{Z}$-module, $g(1)f(1) = 0$ and $f(1)g(1) = 0$, and hence $f(0)g(2) = 0$. Continuing in this way, we get $f(n)g(m) = 0$ for each $n,m \in \mathbb{N}$, and the proof is complete. $\square$

Corollary 2.6. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. Then (HR,α) is reduced if and only if R is α-rigid.*

Let R be a ring, E_{ij} an elementary matrix, n any positive integer, σ an endomorphism of R and I_n the identity matrix in $M_n(R)$. In [9] J. Chen, X. Yang and Y. Zhou introduced the *skew triangular matrix* ring denoted by $T_n(R,\sigma)$

as a set of all triangular matrices with addition pointwise and a new multiplication subject to the condition $E_{ij}r = \sigma^{j-i}(r)E_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$.

The subring of the skew triangular matrices with constant main diagonal is denoted by $S(R,n,\sigma)$. Also, the subring of the skew triangular matrices with constant diagonals is denoted by $T(R,n,\sigma)$ (see [20]). We can denote $A = (a_{ij}) \in T(R,n,\sigma)$ by $(a_0,\ldots,a_{n-1})$. Then $T(R,n,\sigma)$ is a ring with addition pointwise and multiplication given by:
$(a_0,\ldots,a_{n-1})(b_0,\ldots,b_{n-1}) = (a_0b_0, a_0 * b_1 + a_1 * b_0, \ldots, a_0 * b_{n-1} + \cdots + a_{n-1} * b_0)$, with multiplication subject to the condition $a_i * b_j = a_i\sigma^i(b_j)$, for each i and j. On the other hand, there is a ring isomorphism $\varphi : R[x;\sigma]/(x^n) \to T(R,n,\sigma)$, given by $\varphi(\sum\limits_{i=0}^{n-1} a_ix^i) = (a_0,a_1,\ldots,a_{n-1})$, with $a_i \in R$, $0 \leq i \leq n-1$. So $T(R,n,\sigma) \cong R[x;\sigma]/(x^n)$, where $R[x;\sigma]$ is the skew polynomial ring with multiplication subject to the condition $xr = \sigma(r)x$ for each $r \in R$, and (x^n) is the ideal generated by x^n.

Also, we consider the following subrings of $S(R,n,\sigma)$:

$$A(R,n,\sigma) = \left\{ \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{i=1}^{n-j+1} a_jE_{i,i+j-1} + \sum_{j=\lfloor \frac{n}{2} \rfloor+1}^{n} \sum_{i=1}^{n-j+1} a_{i,i+j-1}E_{i,i+j-1} \,|\, a_{i,k} \in R \right\}$$

and

$$B(R,n,\sigma) := \{A + rE_{1k} \,|\, A \in A(R,n,\sigma) \ , \ r \in R \text{ and } n = 2k \geq 4\}.$$

In the special case when $\sigma = id_R$, we use $S(R,n)$, $A(R,n)$, $B(R,n)$ and $T(R,n)$ (see [20]) instead of $S(R,n,\sigma)$, $A(R,n,\sigma)$, $B(R,n,\sigma)$ and $T(R,n,\sigma)$, respectively.

Let R be a ring, α an endomorphism of R, n a positive integer, and $M_n(R)$ the ring of $n \times n$ matrices over R. Then $\overline{\alpha} : M_n(R) \to M_n(R)$, given by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$ is endomorphism of $M_n(R)$.

Proposition 2.7. *[12, Theorems 4.4 and 4.5] Let σ be a rigid endomorphism of R and $A = (a_{ij})$, $B = (b_{ij}) \in T$, where $T \in \{A(R,n,\sigma), B(R,n,\sigma), T(R,n,\sigma)\}$. If $AB = 0$, then $a_{ik}b_{kj} = 0$, for each $1 \leq i,j,k \leq n$.*

By the following proposition, we provide various examples of non-reduced rings which satisfy the α-compatible condition.

Proposition 2.8. *Let σ be a rigid endomorphism of a ring R and α an endomorphism of R. If R is α-rigid, then for every positive integer n, the rings $A(R,n,\sigma)$, $B(R,n,\sigma)$ and $T(R,n,\sigma)$ are $\overline{\alpha}$-compatible rings.*

Proof. Let $A = (a_{ij})$ and $B = (B_{ij}) \in A(R,n,\sigma)$. By Proposition 2.7, we have $AB = 0$ if and only if $a_{ik}b_{kj} = 0$, for each $1 \leq i,j,k \leq n$ if and only if $a_{ik}\alpha(b_{kj}) = 0$ if and only if $A\overline{\alpha}(B) = 0$. Thus $A(R,n,\sigma)$ is $\overline{\alpha}$-compatible. The proof of the other cases are similar. □

3. Diameter and Girth of $\overline{\Gamma}(R)$ and $\overline{\Gamma}((HR,\alpha))$

There is considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. In this section, we examine the preservation and lack thereof of the diameter and girth of the zero-divisor graph of a noncommutative ring under extension to skew Hurwitz series ring. In [30] it has been shown that, for every ring R, any two vertices in $\overline{\Gamma}(R)$ are connected by a path of length at most 3. Note that using the proof of this result in the commutative case, one can establish that for any arbitrary ring R, if there exists a path between two vertices x and y in the directed graph $\Gamma(R)$, then the length of the shortest path between x and y is at most 3. Moreover, in [30] it is shown that for any ring R, if $\overline{\Gamma}(R)$ contains a cycle, then the length of the shortest cycle in $\overline{\Gamma}(R)$ is at most 4.

Let R be a ring with an endomorphism α and I an ideal of R with $\alpha(I) \subseteq I$. Then $\overline{\alpha} : R/I \to R/I$ defined by $\overline{\alpha}(a+I) = \alpha(a)+I$ is an endomorphism of the factor ring R/I. If α is an automorphism and $\alpha(a) \notin I$, for each $a \in R\setminus I$, then $\overline{\alpha}$ is an automorphism. For every nonzero element f of the skew Hurwitz series ring (HR,α), $supp(f)$ is used for the support of f, i.e. $supp(f) = \{i \in \mathbb{N} \mid 0 \neq f(i) \in R\}$.

Theorem 3.1. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. If R is α-compatible, then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}((HR,\alpha))$ is complete.*

Proof. We adapt the proof of [25, Theorem 3.3]. Assume that $\overline{\Gamma}(R)$ be complete, we will show that $\overline{\Gamma}((HR,\alpha))$ is complete. Since R is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, by [2, Theorem 5] we deduce that $Z(R)^2 = 0$ and $Z(R)$ is an ideal of R. Now, since R is α-compatible, it is easy to show that $\overline{R} := \frac{R}{Z(R)}$ is $\overline{\alpha}$-rigid. Now, we claim that $Z((HR,\alpha)) \subseteq (H(Z(R)),\alpha)$. Suppose towards a contradiction that $f \in Z((HR,\alpha)) \setminus (H(Z(R)),\alpha)$. Then there exists a nonzero element $g \in (HR,\alpha)$ such that $fg = 0$. We will prove that $\overline{g} \neq \overline{0}$. Assume on the contrary that $\overline{g} = \overline{0}$, or, equivalently, that $g \in (H(Z(R)),\alpha)$. Now, we have the following two cases.

Case (1) Let $f(n) \notin Z(R)$, for all $n \in supp(f)$. Let n_0 and m_0 denote the minimal elements of $supp(f)$ and $supp(g)$, respectively. Hence, we have:

$$0=(fg)(n_0+m_0)=\sum_{i+j=n_0+m_0}\left(\begin{array}{c}i+j\\ i\end{array}\right)f(i)\alpha^i(g(j))$$
$$=\left(\begin{array}{c}n_0+m_0\\ n_0\end{array}\right)f(n_0)\alpha^{n_0}(g(m_0)).$$

Since R is torsion free as a $\mathbb{Z}$-module, we have $f(n_0)\alpha^{n_0}(g(m_0))=0$. Thus the α-compatibility of R implies that $f(n_0)g(m_0)=0$. Therefore $f(n_0)\in Z(R)$, which is a contradiction.

Case (2) Let $\mathfrak{D}:=\{n\in supp(f)\mid f(n)\in Z(R)\}$ be nonempty. Set:

$$h(n):=\begin{cases} f(n) & n\in\mathfrak{D}\\ 0 & n\notin\mathfrak{D}\end{cases} \text{ and } k(n):=\begin{cases} f(n) & n\notin\mathfrak{D}\\ 0 & n\in\mathfrak{D}.\end{cases}$$

We obtain maps $h,k:S\to R$ with $supp(h)=\mathfrak{D}$ and $supp(k)=\mathfrak{D}^c\cap supp(f)$. Since $g\in(H(Z(R)),\alpha)$ and $Z(R)^2=0$, we have $h(n)g(m)=0$ for each $n,m\in\mathbb{N}$. Now, the α-compatibility of R implies that $hg=0$. Therefore $kg=0$. By a similar argument, there exist $n_0\in supp(k)$ and $m_0\in supp(g)$ such that $0=(kg)(n_0+m_0)=k(n_0)\alpha^{n_0}(g(m_0))$. Using the α-compatibility of R, we find that $k(n_0)g(m_0)=0$. Therefore $k(n_0)=f(n_0)\in Z(R)$, which is a contradiction.
Therefore we conclude that $\overline{g}\neq\overline{0}$. Since $\overline{R}$ is $\overline{\alpha}$-rigid and $\overline{fg}=\overline{0}$, we obtain $\overline{f(n)g(m)}=\overline{0}$, for all $n,m\in\mathbb{N}$, by Lemma 2.5. On the other hand, $\overline{f},\overline{g}\neq 0$, thus there exist $n\in suup(\overline{f})$ and $m\in suup(\overline{g})$ such that $\overline{f(n)g(m)}=\overline{0}$, contrary to the fact that $\overline{R}$ has no zero divisors $\neq 0$. Therefore $Z((HR,\alpha))\subseteq(H(Z(R)),\alpha)$. Now, assume that f, g are two distinct elements in $Z^*((HR,\alpha))$. Consequently $f(n),g(n)\in Z(R)$ for all $n\in\mathbb{N}$. Since $Z(R)^2=0$ and R is α-compatible, $a\alpha^n(b)=0$ for every $n\in\mathbb{N}$ and $a,b\in Z^*(R)$. Therefore

$$fg(n)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\ k\end{array}\right)f(k)\alpha^k(g(n-k))=0$$

for every $n\in\mathbb{N}$. So $\overline{\Gamma}((HR,\alpha))$ is complete. The converse follows directly from the fact that $\overline{\Gamma}(R)$ is an induced subgraph of $\overline{\Gamma}((HR,\alpha))$, and the proof is complete. □

Corollary 3.2. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module. Then $\overline{\Gamma}(R)$ is complete if and only if $\overline{\Gamma}(HR)$ is complete.*

Remark 3.3. If $R\cong\mathbb{Z}_2\times\mathbb{Z}_2$, then $\overline{\Gamma}(R)=1$. Now, let $\alpha:R\to R$ be the endomorphism of R defined by $\alpha(a,b)=(b,a)$, for all $(a,b)\in R$. Suppose that $f=h_{-(0,1)}+h_{(0,1)}h_n^{'}$ and $g=h_{(0,1)}+h_{(0,1)}h_n^{'}$ in (HR,α), where $n\in\mathbb{N}\setminus\{0\}$. Then $fg\neq 0$, but $h_{(1,0)}f=h_{(1,0)}g=0$. So $f-h_{(1,0)}-g$ is a path in (HR,α) and thus $diam(\overline{\Gamma}((HR,\alpha)))\geq 2$.

Now, we provide the following example to illustrate Theorem 3.1.

Example 3.4. Assume that D is a domain. Put:

$$R := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in D \right\}.$$

Suppose that $\alpha : D \to D$ is a monomorphism of D. Then $\overline{\Gamma}(R)$ is complete, since $Z(R)^* = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in D^* \right\}$. On the other hand, it is easy to show that R is $\overline{\alpha}$-compatible, where $\overline{\alpha} : R \to R$ is an endomorphism of R given by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. Therefore $\overline{\Gamma}((HR, \overline{\alpha}))$ is complete, by Theorem 3.1.

Remark 3.5. For a commutative ring R with identity, the collection of zero-divisors $Z(R)$ of R is the set-theoretic union of prime ideals $\bigcup_{i \in \Lambda} \mathcal{P}_i$. We will also assume that these primes are maximal with respect to being contained in $Z(R)$. So if $diam(\Gamma(R)) \leq 2$ and Λ is a finite set (in particular if R is Noetherian), in view of [6, Corollary 3.5], $|\Lambda| \leq 2$.

Proposition 3.6. *[6, Proposition 3.6] Let R be a commutative ring such that $diam(\Gamma(R)) = 2$. If $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ with $\mathcal{P}_1$ and $\mathcal{P}_2$ distinct maximal primes in $Z(R)$, then $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$ (in particular, for all $p_1 \in \mathcal{P}_1$ and $p_2 \in \mathcal{P}_2$, $p_1 p_2 = 0$).*

Theorem 3.7. *Let R be a commutative reduced ring which is torsion free as a $\mathbb{Z}$-module. If $diam(\Gamma(R)) = 2$, then $diam(\Gamma(HR)) = 2$.*

Proof. By Remark 3.5, either $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ union of precisely two maximal primes in $Z(R)$, or $Z(R) = \mathcal{P}$ is a prime ideal.

Case (1) Suppose that $Z(R) = \mathcal{P}_1 \cup \mathcal{P}_2$ is the union of precisely two maximal primes in $Z(R)$. Let f and g are two distinct elements in $Z^*(HR)$. By Lemma 2.5, $f(n), g(n) \in Z(R)$ for all $n \in \mathbb{N}$. Then it is necessary for f (and hence g) to be contained in $H\mathcal{P}_1$ or $H\mathcal{P}_2$. Because otherwise there exist $f(n) \in \mathcal{P}_1 \setminus \mathcal{P}_2$ and $f(m) \in \mathcal{P}_2 \setminus \mathcal{P}_1$ such that $f(n)r = 0$ and $f(m)r = 0$ for some nonzero element r of R and $n, m \in \mathbb{N}$. Thus $r \in \mathcal{P}_1 \cap \mathcal{P}_2$, contrary to the fact that $\mathcal{P}_1 \cap \mathcal{P}_2 = \{0\}$, by Proposition 3.6. So, we have two cases. Firstly, suppose that $f \in H\mathcal{P}_1$ and $g \in H\mathcal{P}_2$. Then, by Proposition 3.6, $f(n)g(m) = 0$, for all $n, m \in \mathbb{N}$. Hence $fg = 0$. Now, consider the case that $f, g \in H\mathcal{P}_1$. Then any element of $\mathcal{P}_2$ suffices as a mutual annihilator. Thus $diam(\Gamma(HR)) = 2$.

Case (2) Assume that $Z(R) = \mathcal{P}$ is a prime ideal. By [14, Theorem 82], $\mathcal{P}$ is annihilated by a single element (say z). Suppose that f, g are zero-divisors. If $fg = 0$, we are done. If $fg \neq 0$, then z is a mutual annihilator of f and g, by Lemma 2.5. □

Proposition 3.8. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module. If R is α-compatible and $diam(\overline{\Gamma}((HR,\alpha))) = 2$, then $diam(\overline{\Gamma}(R)) = 2$.*

Proof. It is easy to show that $diam(\overline{\Gamma}(R)) \leq diam(\overline{\Gamma}((HR,\alpha)))$. The result now follows from Theorem 3.1. □

McCoy [24, Theorem 2] proved that if two nonzero polynomials annihilate each other over a commutative ring then each polynomial has a non-zero annihilator in the base ring. But Weiner [33] showed that this theorem fails in noncommutative rings. According to Nielsen [26], a ring R is called (linearly) *right McCoy* if the equation $f(x)g(x) = 0$, for (linear) polynomials $f(x), g(x) \in R[x] \setminus \{0\}$, implies that there exists $0 \neq c \in R$ such that $f(x)c = 0$. Left McCoy rings are defined similarly. If a ring is both left McCoy and right McCoy, it is called a McCoy ring. In [26, Theorem 2], Nielsen showed that reversible rings are McCoy. It is obvious that every commutative ring is reversible. Thus, reversible rings provide a sort of bridge between commutative and noncommutative ring theory.

Now, we apply the concept of McCoy rings to skew Hurwitz series rings over non-commutative rings, and introduce McCoy of skew Hurwitz series type rings. Recall that a ring R is called *right McCoy of skew Hurwitz series type* (or simply, α-SMHS ring), if whenever elements $f, g \in (HR,\alpha) \setminus \{0\}$ satisfy $fg = 0$, then there exists $0 \neq r \in R$ such that $fr = 0$. Left α-SMHS rings is defined similarly. If R are both a left and right α-SMHS ring, then we say that R is α-SMHS ring. By Lemma 2.5, it is easy to show that every α-rigid ring R which is torsion free as a $\mathbb{Z}$-module is an α-SMHS ring. In Theorem 3.9 and Proposition 3.10, we provide several methods of constructing α-SMHS rings which are not reduced.

Let R be a ring, and α, σ are endomorphisms of R. It is easy to verify that the map $\widetilde{\sigma} : (HR,\alpha) \to (HR,\alpha)$, given by $\widetilde{\sigma}(f) = \sigma \circ f$, is an endomorphism of (HR,α).

Theorem 3.9. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module and α, σ are endomorphisms of R with $\sigma \circ \alpha = \alpha \circ \sigma$. If R is an α-rigid ring, then for every positive integer n, the rings $S(R,n,\sigma)$, $A(R,n,\sigma)$, $B(R,n,\sigma)$ and $T(R,n,\sigma)$ are $\overline{\alpha}$-SMHS ring.*

Proof. We only prove $S(R,n,\sigma)$ is right $\overline{\alpha}$-SMHS, because the proofs of the other cases are similar.

First consider the map $\phi : (H(S(R,n,\sigma)),\overline{\alpha}) \to S((HR,\alpha),n,\widetilde{\sigma})$, given by $\phi(f) = (f_{ij})$, where $f_{ij}(n) = (f(n))_{ij}$ for all $n \in \mathbb{N}$ and the $(f(n))_{ij}$ is the (i,j)-th entry of $f(n)$. It is easy to show that ϕ is an isomorphism.
Suppose that $f, g \in (H(S(R,n,\sigma)),\overline{\alpha}) \setminus \{0\}$ satisfy $fg = 0$. Then $(f_{ij})(g_{ij}) =$

0 for $1 \leq i, j \leq n$. If $f_{11} = 0$, clearly $(f_{ij})E_{1n} = 0$. Hence $fh_{E_{1n}} = 0$. Thus $S(R, n, \overline{\sigma})$ is $\overline{\alpha}$-SMHS. Now, suppose $f_{11} \neq 0$. Since $\phi(g) \neq 0$, there exists a non-zero (k,l)-entry of $\phi(g)$. Assume $g_{kl} \neq 0$ with k maximal. Considering the (k,l)-entry of $\phi(f)\phi(g)$, we have $f_{11}g_{kl} = 0$. Since R is α-rigid, there exists $0 \neq r \in R$ such that $f_{11}h_r = 0$, by Lemma 2.5. Hence $fh_D = 0$, where $D = rE_{1n} \in S(R, n, \sigma)$. Therefore $S(R, n, \sigma)$ is an $\overline{\alpha}$-SMHS ring. $\square$

Proposition 3.10. *Let R be a ring and α, σ are endomorphisms of R such that $\sigma \circ \alpha = \alpha \circ \sigma$. Define a subring $V(R, \sigma)$ of $T_6(R, \sigma)$ as follows:*

$$V(R,\sigma) := \left\{ \begin{pmatrix} a & d & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & e & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & f \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix} \mid a,b,c,d,e,f \in R \right\}.$$

Then R is a right (resp. left) α-SMHS ring if and only if $V(R, \sigma)$ is a right (resp. left) $\overline{\alpha}$-SMHS ring.

Proof. We only prove the case when R is right α-SMHS and the other case is similar. First consider the map $\phi : (H(V(R,\sigma)), \overline{\alpha}) \to V((HR, \alpha), \widetilde{\sigma})$, given by $\phi(f) = (f_{ij})$, where $f_{ij}(n) = (f(n))_{ij}$ for all $n \in \mathbb{N}$ and the $(f(n))_{ij}$ is the (i, j)-th entry of $f(n)$. It is easy to show that ϕ is an isomorphism.
Suppose that R is right α-SMHS. Assume that f, g are non-zero elements of $(H(V(R,\sigma)), \overline{\alpha})$ such that $fg = 0$. Then $(f_{ij})(g_{ij}) = 0$ for $1 \leq i, j \leq 6$. So we have the following equations:

$$f_{11}g_{11} = 0; f_{22}g_{22} = 0; f_{33}g_{33} = 0;$$

$$f_{11}g_{12} + f_{12}\widetilde{\sigma}(g_{22}) = 0; f_{33}g_{34} + f_{34}\widetilde{\sigma}(g_{11}) = 0; f_{22}g_{56} + f_{56}\widetilde{\sigma}(g_{33}) = 0.$$

Let $f_{ii} = 0$ for some $1 \leq i \leq 3$, then we can choose $A = E_{12}$ if $i = 1$, $A = E_{56}$ if $i = 2$ and $A = E_{34}$ if $i = 3$. Clearly $(f_{ij})A = 0$. Hence $fh_A = 0$. Next suppose that $f_{ii} \neq 0$ for all $1 \leq i \leq 3$.

Case 1) Let $g_{ii} \neq 0$ for some $1 \leq i \leq 3$.

Suppose that $i = 3$. Since $f_{33}g_{33} = 0$ and R is right α-SMHS, so there exists $0 \neq r \in R$ such that $f_{33}h_r = 0$. Hence $fh_D = 0$, where $D = rE_{34} \in V(R, \sigma)$.

Case 2) Let $g_{ii} = 0$ for every $1 \leq i \leq 3$.

Since $g \neq 0$, we may assume without loss of generality $g_{12} \neq 0$. Since $f_{11}g_{12} = 0$ and R is right α-SMHS, so there exists $0 \neq r \in R$ such that $f_{11}h_r = 0$. Hence $fh_D = 0$, where $D = rE_{12} \in V(R, \sigma)$. Therefore, $V(R, \sigma)$ is right $\overline{\alpha}$-SMHS. For the forward implication, let f and g be non-zero elements in

(HR,α) such that $fg=0$. Assume that $F(n)=f(n)I_6$ and $G(n)=g(n)I_6$, for all $n\in\mathbb{N}$. Therefore $FG=0$. Since $V(R,\sigma)$ is right $\overline{\alpha}$-SMHS, there exists $0\neq A=(a_{ij})\in V(R,\sigma)$ such that $Fh_A=0$. Since A is non-zero, there exists non-zero a_{ij} for some $1\leq i,j\leq 6$ and $fh_{a_{ij}}=0$. Hence R is α-SMHS and the result follows. □

The following example shows that the $n\times n$ (upper triangular) matrix ring over a ring is not an α-SMHS ring.

Example 3.11. Let R be a ring and α be a monomorphism of R. We show that the 2-by-2 matrices over R is not $\overline{\alpha}$-SMHS. Let $f=h_{E_{11}}+h_{(E_{11}-E_{12})}h_n'$ and $g=h_{E_{22}}+h_{(E_{12}+E_{22})}h_n'$ be elements of $(H(M_2(R)),\alpha)$, where $n\in\mathbb{N}\setminus\{0\}$. Then $fg=0$. But $fh_A=0$ implies $A=0\in M_2(R)$. So $M_2(R)$ is not right $\overline{\alpha}$-SMHS, and consequently $n\times n$ (upper triangular) matrix rings over a ring are not α-SMHS.

Proposition 3.12. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module. If R is an α-SMHS ring, α-compatible and for some $n\in\mathbb{Z}$ with $n>2$, $(Z(R))^n=0$ and $(Z(R))^{n-1}\neq 0$, then:*

$$diam(\overline{\Gamma}(R))=diam(\overline{\Gamma}((HR,\alpha)))=2.$$

Proof. Since the ring R is not isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$ and $(Z(R))^2\neq 0$, by [2, Theorem 5], $\overline{\Gamma}(R)$ is not complete. Hence, there exist distinct $a,b\in Z(R)$ with $ab\neq 0$ and $ba\neq 0$. On the other hand, since $(Z(R))^{n-1}\neq 0$, there exist $d_1,d_2,\ldots,d_n\in Z(R)$ with $d=\prod_{i=1}^{n-1}d_i\neq 0$. Therefore, $ad=0=bd$, since $(Z(R))^n=0$. So $a-d-b$ is a path in R and hence $diam(\overline{\Gamma}(R))=2$. Now, it is sufficient to prove that $diam(\overline{\Gamma}((HR,\alpha)))=2$. Let f and g be two distinct element in $Z^*((HR,\alpha))$. Since R is α-SMHS, we get $f(n),g(n)\in Z(R)$ for all $n\in\mathbb{N}$. Hence the α-compatibility of R yields either $f-g$ or $f-h_d-g$. Therefore $diam(\overline{\Gamma}((HR,\alpha)))=2$, and the proof is complete. □

Proposition 3.13. *Let R be a ring which is not a domain and α an endomorphism of R. Assume that R is α-compatible. Then $gr(\overline{\Gamma}((HR,\alpha)))$ is either 3 or 4. In particular, if R is not reduced, then $gr(\overline{\Gamma}((HR,\alpha)))=3$.*

Proof. Let $ab=0$ for distinct elements $a,b\in Z^*(R)$. Using the α-compatibility of R, we find that $a\alpha^n(b)=0$, for all $n\in\mathbb{N}\setminus\{0\}$. Hence $h_a-h_b-h_ah_n'-h_bh_n'-h_a$ is a 4-cycle in (HR,α). Let $a^2=0$ for some $a\in Z^*(R)$. Then the α-compatibility of R, yields $h_a-h_ah_n'-h_ah_{2n}'-h_a$ is a 3-cycle in (HR,α), for all $n\in\mathbb{N}\setminus\{0\}$. □

The following and its proof are directly adapted from [25, Theorem 3.22].

Theorem 3.14. *Let R be a ring such that it is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. If R is α-rigid and $\overline{\Gamma}(R)$ contains a cycle, then $gr(\overline{\Gamma}(R)) = gr(\overline{\Gamma}((HR,\alpha)))$.*

Proof. If $Z^*(R) = \emptyset$, then $gr(\overline{\Gamma}(R)) = \infty = gr(\overline{\Gamma}((HR,\alpha)))$. So we may assume $Z^*(R) \neq \emptyset$. Since the graph $\overline{\Gamma}(R)$ is an induced subgraph of $\overline{\Gamma}((HR,\alpha))$, we have that $gr(\overline{\Gamma}(R)) \geq gr(\overline{\Gamma}((HR,\alpha)))$. Also, by Proposition 3.13, $gr(\overline{\Gamma}((HR,\alpha))) \leq 4$. Furthermore, since $\overline{\Gamma}(R)$ contains a cycle, by [30], $gr(\overline{\Gamma}(R)) \leq 4$. Therefore it suffices to show that $gr(\overline{\Gamma}(R)) = 3$, whenever $gr(\overline{\Gamma}((HR,\alpha))) = 3$. Hence suppose that $f - g - h - f$ is a cycle in (HR,α). Since $fg = gh = hf = 0$ thus by Lemma 2.5, we have $f(m)g(n) = g(n)h(k) = h(k)f(m) = 0$ for all $m,n,k \in \mathbb{N}$. We may assume that $f(m_0)$, $g(n_0)$ and $h(k_0)$ are non-zero elements in R. Therefore $f(m_0)g(n_0) = g(n_0)h(k_0) = h(k_0)f(m_0) = 0$. Moreover, the elements $f(m_0)$, $g(n_0)$ and $h(k_0)$ are distinct by Lemma 2.4, since R is reduced. Now consider the cycle $f(m_0) - g(n_0) - h(k_0) - f(m_0)$ of length three in $\overline{\Gamma}(R)$. Therefore $gr(\overline{\Gamma}(R)) = 3$, and hence $gr(\overline{\Gamma}(R)) = gr(\overline{\Gamma}((HR,\alpha)))$, and the result follows. □

A complete characterization for the girth of $gr(\Gamma(R[x]))$ and $gr(\Gamma(R[[x]]))$ in terms of $gr(\Gamma(R))$ is given in [5, Theorem 3.2]. In the following we explain Theorem 3.2 in [5] in the context of skew Hurwitz series extension rings.

Corollary 3.15. *Let R be a ring which is torsion free as a $\mathbb{Z}$-module and α an endomorphism of R. Assume that R is α-compatible.*
(1) Suppose that $\overline{\Gamma}(R)$ is nonempty with $gr(\overline{\Gamma}(R)) = \infty$.

(i) *If R is reduced, then $gr(\overline{\Gamma}((HR,\alpha))) = 4$;*

(ii) *If R is not reduced, then $gr(\overline{\Gamma}((HR,\alpha))) = 3$.*

(2) If $gr(\overline{\Gamma}(R)) = 3$, then $gr(\overline{\Gamma}((HR,\alpha))) = 3$.
(3) Suppose that $gr(\overline{\Gamma}(R)) = 4$.

(i) *If R is reduced, then $gr(\overline{\Gamma}((HR,\alpha))) = 4$;*

(ii) *If R is not reduced, then $gr(\overline{\Gamma}((HR,\alpha))) = 3$.*

Proof. We have already observed in Proposition 3.13 that $gr(\overline{\Gamma}((HR,\alpha)))) = 3$ if R is not reduced. Thus (1) (*ii*) and (3) (*ii*) hold. By using the proof of Theorem 3.14, if R is reduced and $gr(\overline{\Gamma}((HR,\alpha))) = 3$, then $gr(\overline{\Gamma}(R)) = 3$. Now, since $gr(\overline{\Gamma}((HR,\alpha))) \leq 4$, by Proposition 3.13, and thus (1) (*i*) and (3) (*i*) hold. Clearly (2) holds since $gr(\overline{\Gamma}(R)) \geq gr(\overline{\Gamma}((HR,\alpha)))$. □

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KAMAL PAYKAN
Department of Basic Sciences
Garmsar Branch
Islamic Azad University
Garmsar, IRAN
e-mail: k.paykan@gmail.com

HOSSEIN DANESHMAND
Shahid Shamsipour Colloge
Technical and Vocational University
Tehran, IRAN
e-mail: danesh498@yahoo.com